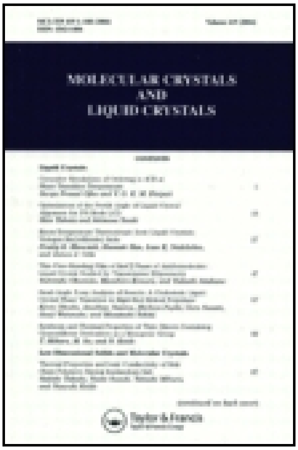

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The Torque Exerted on a Narrow Slab of a Nematic Liquid Crystal in a Magnetic Field

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THE TORQUE EXERTED ON A NARROW SLAB OF A NEMATIC LIQUID CRYSTAL IN A MAGNETIC FIELD

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The hydrodynamic equations for a narrow slab when the state of nematic liquid crystal is deformed slightly by a magnetic field are solved. Then the torque exerted on the torsional oscillator in which the slab is contained is calculated, and the changes in the resonant frequency as a function of the magnetic field are determined.

Keywords: nematic liquid crystals; hydrodynamics; Freedericksz transition

INTRODUCTION

One can use liquid crystals (LC) in a variety of applications because external perturbation can cause significant changes in the macroscopic properties of the LC system. Both electric and magnetic fields can be used to induce these changes. Special surface treatment can be used in LC devices to force specific orientation of the director. The effects of magnetic field on LC molecules can be caused by moving electric charges, since permanent magnetic dipoles are produced by electrons moving about atoms and the field will tend to align the molecules with or against it [1].

The competition between orientation produced by the surface anchoring and by the electric or magnetic effects aligns the molecules in a certain direction. At first, as the fields increase in magnitudes no change in alignment occurs; however, at a threshold field deformation occurs where the director changes its orientation from one molecule to the next. The occurrence of such a change from an aligned to a deformed state is called a *Frederiks transition*. The Frederiks transition in nematic liquid crystal films has long been the subject of experimental and theoretical studies that have elucidated the nature of elastic deformations in liquid crystal and

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advanced our understanding of the anchoring of their molecules at solid surfaces [1,2].

Since most observed processes in nematic liquid crystal involve small reorientation, we will continue our discussion on the small reorientation, θ, which denotes the average direction of liquid crystal alignment (see below). In this case, using the so-called hard boundary condition or strongly anchored nematic liquid crystal, the director axis is not nearly perturbed at the boundary of the slab. The substrates of the slab are placed at $z = \pm d/2$, where d is the width of the slab and the z axis is perpendicular to it. In this article we consider only a slab of the nematic liquid crystal, which is located in a torsional oscillator, and a magnetic field B normal to the slab is applied (Figure 1).

Since molecules of the nematic liquid crystal are anisotropy, as we said, an external magnetic field can induce an alignment or ordering in the isotropic phase, or a realignment of the molecules in the ordered phase. These result in a change in the torque on a slab of nematic liquid crystal at the Frederiks transition, which is the subject of this article for the calculation.

For nematic liquid crystal, two commonly used alignments are the so-called homotropic and planner alignments. The surfacetant molecules are perpendicular or parallel to the wall for the homotropic or planner alignments, respectively. In this article the planner alignment will be considered.

Olevy recently studies the Frederiks transition in dispersions of liquid crystal droplets analytically [3]. He showed that the transition threshold field depends on the liquid crystal volume fraction and the spacial arrangement of the droplets. One of the probes to justify this theory is to calculate the change in the torque that exerted on the film of polymer-dispersed

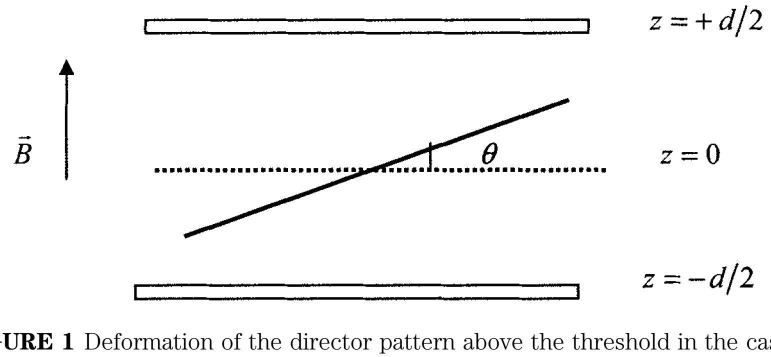

FIGURE 1 Deformation of the director pattern above the threshold in the case of splay mode.

liquid crystal. This problem is under our consideration and will be published elsewhere.

In this article, first we briefly discuss the Euler equations for θ(z) and ϕ(z), where $[\bar{n} = (\theta(z), \phi(z))]$ is the average direction of liquid crystal alignment and their variational solutions. We next calculate the hydrodynamic equations and obtain the components of velocities. Finally, we compute the torque exerted on the slab and give some concluding remarks.

EULER EQUATIONS AND VARIATIONAL SOLUTION

Consider a slab of nematic liquid crystal inside a torsional oscillator of width d (Figure 1) placed in a magnetic field B. We shall assume that the orbital texture is not affected by the flow associated with the motion of the torsional oscillator.

At any given field, the stable equilibrium state can be found by minimizing the total free energy with respect to variations in the director pattern. Let θ(z) be the tilt angle between the director and the xy plane. The director is then given by $\vec{n} = [0, \cos\theta(z), \sin\theta(z)]$. The total free energy is written as [4]

$$f = f_0 + \frac{1}{2}\left[(K_1\cos^2\theta + K_3\sin^2\theta)\left(\frac{d\theta}{dz}\right)^2 - \frac{\chi_\alpha}{\mu_0}B^2 sin^2\theta\right], \qquad (1)$$

where f_0 is the free-energy density of the uniform director pattern, and K_1 and K_3 are elastic constants.

The value of θ(z) will be such that f is a minimum with $\theta \cong 0$ for $z = \pm d/2$. This condition leads to the Euler-Lagrange equation, which can be written directly as

$$(K_1\, cos^2\theta + K_3\, sin^2\theta)\left(\frac{\partial\theta}{\partial z}\right)^2 + \mu_0^{-1}\chi_\alpha B^2\, sin^2\theta = C. \qquad (2)$$

The constant C can be determined from the fact that θ(z) has a maximum value of θ_{m} for $z = 0$. Consequently, at this position we have $\frac{\partial\theta}{\partial z} = 0$ and $C = \mu_0^{-1}\chi_\alpha B^2 sin^2\theta_m$. By using one constant approximation, i.e., $K_1 = K_3 = K$ and small values of θ(z), one can write Equation (2) as

$$\frac{dz}{d\theta} = \frac{1}{B}\left[\frac{\mu_0}{\chi_\alpha}\frac{K}{\theta_m^2 - \theta^2}\right]^{1/2}. \qquad (3)$$

The solution of Equation (3) is

$$\theta(z) = \theta_m \cos\left(\frac{\pi}{d}z\right), \qquad (4)$$

where θ_m is determined variationally.

To calculate the threshold field we return to Equation (3). This can readily be integrated from $\theta = \theta_{\rm m}$ to $\theta \cong 0$ and z = 0 to $z = \pm d/2$, giving the threshold field as

$$B_c = \frac{\pi}{d}\left(\frac{\mu_0 K}{\chi_\alpha}\right)^{1/2}. \tag{5}$$

To calculate θ_m, we first calculate the total free energy of the slab, $(\langle f\rangle = \int f dz)$. This total free energy can be minimized with respect to the amplitude θ_m for a fixed value of applied magnetic field B. This gives

$$\langle f\rangle \equiv \frac{\chi_\alpha d}{2\mu_0}\left(\frac{B^2}{8}\theta_m^4 - \frac{B^2 - B_c^2}{2}\theta_m^2\right) \tag{6}$$

and

$$\theta_m^2 = 2\left[1 - \left(\frac{B_c}{B}\right)^2\right]$$

or

$$\theta_m = 0, \tag{7}$$

where $\theta_m = 0$ represents the uniform director pattern.

THE HYDRODYNAMIC EQUATIONS OF A NARROW SLAB OF NEMATIC LIQUID CRYSTAL IN TORSIONAL OSCILLATOR

We now proceed to solve the hydrodynamic equations of a narrow slab of nematic in a torsional with small oscillation amplitudes for obtaining the velocity field components. We assume that the total density of the fluid remains constant during the oscillator motion. Then the conservation of mass requires that

$$\vec{\nabla}\cdot(\rho\vec{v}) = -\frac{\partial\rho}{\partial t}. \tag{8}$$

Next we write to the equation of motion. For this purpose consider a small volume in the liquid. Then we may write

$$\rho\frac{d\vec{v}}{dt} = \vec{f}, \tag{9}$$

where $\vec{f}$ is the force per unit volume. From the definition of partial derivatives we may write

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}\,.\,\vec{\nabla}). \tag{10}$$

Therefore,

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\,.\,\vec{\nabla})\,\vec{v} = \vec{f}\,. \tag{11}$$

There may be various contributions to the total force $\vec{f}$: first, the hydrodynamic forces, $-\vec{\nabla} P$, which is due to the pressure and the viscous term $\vec{f}_{visc}$; second, the magnetic force, $f_m = \chi_\alpha B^2 \sin\theta \frac{\partial\theta}{\partial z}\hat{k}$; and finally the gravitational force, which can be ignored for the narrow slab. Since the oscillator amplitude is supposed to be small, we may use the hydrostatic condition on the pressure. On the other hand, at Frederiks transition there is competition between distortion and magnetic forces, and we may approximately take the pressure to be constant [1], hence Equation (11) becomes

$$\rho\left[\left(\frac{\partial \vec{v}}{\partial t}\right) + (\vec{v}\,.\,\vec{\nabla})\,\vec{v}\right] = \vec{f}_{visc}. \tag{12}$$

The force per unit area being called the stress, hence Equation (12) can be rewritten as

$$\rho\left[\left(\frac{\partial v_\alpha}{\partial t}\right) + v_\beta \frac{\partial v_\alpha}{\partial x_\beta}\right] = \frac{\partial \sigma_{\beta\alpha}}{\partial x_\beta}, \tag{13}$$

where the elements of stress tensor $\sigma_{\beta\alpha}$ are defined as [2]

$$\begin{aligned}\sigma_{\alpha\beta} = {} & \alpha_4 A_{\alpha\beta} + \alpha_1 n_\alpha n_\beta n_\rho n_\mu A_{\mu\rho} + \alpha_2 n_\alpha N_\beta + \alpha_3 n_\beta N_\alpha \\ & + \alpha_5 n_\alpha n_\mu A_{\mu\beta} + \alpha_6 n_\beta n_\mu A_{\mu\alpha.}\end{aligned} \tag{14}$$

There are six coefficients with the dimension of a viscosity that are often called Leslie coefficients, where

$$\alpha_6 = \alpha_2 + \alpha_3 + \alpha_5. \tag{15}$$

Hence only five coefficients are independent, also

$$N_\alpha = \frac{dn_\alpha}{dt} - W_{\alpha\beta} n_\beta, \tag{16}$$

$$A_{\alpha\beta} = A_{\beta\alpha} = \frac{1}{2}\left(\frac{\partial v_\beta}{\partial x_\alpha} + \frac{\partial v_\alpha}{\partial x_\beta}\right), \tag{17}$$

$$W_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial v_\beta}{\partial x_\alpha} - \frac{\partial v_\alpha}{\partial x_\beta}\right). \tag{18}$$

To solve the hydrodynamic equation of the planer texture, we assume velocity field of the form [4]

$$\vec{v} = [v_x(y,z)\hat{i} + v_y(x,z)\hat{j} + v_z(x,z)\hat{k}]\exp(i\omega_0 t). \tag{19}$$

On the other hand, the liquid crystal is an incompressible fluid, so using Equation (8) we have

$$\vec{\nabla}\,.\,\vec{v} = 0$$

or

$$\frac{\partial v_z}{\partial z} = 0. \tag{20}$$

By substituting the above equation into the velocity field in Equation (19), we have

$$\vec{v} = [v_x(y,z)\hat{i} + v_y(x,z)\hat{j} + v_z(x)\hat{k}]\exp(i\omega_0 t). \tag{21}$$

By using Equations (14) and (21) in (13) we obtain the following equations for the planar texture:

$$\begin{aligned}\rho i\omega_0 v_x + \rho v_y\frac{\partial v_x}{\partial y} + \rho v_z\frac{\partial v_x}{\partial z} &= \frac{1}{2}(\alpha_4 + (\alpha_2+\alpha_5)\cos^2\theta)\frac{\partial^2 v_x}{\partial y^2}\\ &+ \frac{1}{2}(\alpha_4 + (\alpha_2+\alpha_5)\sin^2\theta)\frac{\partial^2 v_x}{\partial z^2} + \frac{1}{2}(\alpha_2+\alpha_5)\sin\theta\cos\theta\frac{\partial^2 v_x}{\partial y\partial z}\\ &+ \frac{1}{2}(\alpha_5-\alpha_2)\sin\theta\cos\theta\frac{\partial^2 v_y}{\partial x\partial z} + \frac{1}{2}\frac{\partial\sin^2\theta}{\partial z}\left[(\alpha_2+\alpha_5)\frac{\partial v_x}{\partial z} + (\alpha_5-\alpha_2)\frac{\partial v_z}{\partial x}\right]\\ &+ \frac{\partial(\sin\theta\cos\theta)}{\partial z}\left[(\alpha_5+\alpha_2)\frac{\partial v_x}{\partial y} + (\alpha_5-\alpha_2)\frac{\partial v_y}{\partial x}\right],\end{aligned} \tag{22}$$

$$\begin{aligned}\rho i\omega_0 v_y + \rho v_x\frac{\partial v_y}{\partial x} + \rho v_z\frac{\partial v_y}{\partial z} &= \frac{1}{2}(\alpha_4 + (\alpha_2+\alpha_5)\cos^2\theta)\frac{\partial^2 v_y}{\partial x^2}\\ &+ \frac{1}{2}(\alpha_2+\alpha_4+\alpha_5+2\alpha_1\sin^2\theta\cos^2\theta)\frac{\partial^2 v_y}{\partial z^2} + \frac{1}{2}(\alpha_2+\alpha_5)\sin\theta\cos\theta\frac{\partial^2 v_z}{\partial x^2}\\ &+ \alpha_1\frac{\partial(\sin^2\theta\cos^2\theta)}{\partial z}\frac{\partial v_y}{\partial z},\end{aligned} \tag{23}$$

$$\begin{aligned}\rho i\omega_0 v_z + \rho v_x\frac{\partial v_z}{\partial x} &= \frac{1}{2}(\alpha_4 + (\alpha_2+\alpha_5)\sin^2\theta)\frac{\partial^2 v_z}{\partial x^2} + \frac{1}{2}(\alpha_2+\alpha_5)\sin\theta\cos\theta\frac{\partial^2 v_y}{\partial x^2}\\ &+ (\alpha_1\sin^3\theta\cos\theta + \alpha_5\sin\theta\cos\theta)\frac{\partial^2 v_y}{\partial z^2} + \frac{\partial(\alpha_1\sin^3\theta\cos\theta + \alpha_5\sin\theta\cos\theta)}{\partial z}\frac{\partial v_y}{\partial z}.\end{aligned} \tag{24}$$

As we mentioned previously, we are considering the state of nematic LC in a torsional oscillator which is deformed slightly. On the basis of this assumption and the symmetry of the problem, it is possible to use the following approximations for the velocity field components:

$$v_x(y,z) = \sum_{n=0}^{\infty} v_{nx}(y,z) \quad \theta_m^n = \sum_{n=0}^{\infty} f_n(y) g_n(z) \theta_m^n, \tag{25}$$

$$v_y(x,z) = \sum_{n=0}^{\infty} v_{ny}(x,z) \quad \theta_m^n = \sum_{n=0}^{\infty} L_n(x) M_n(z) \theta_m^n, \tag{26}$$

$$v_z(x) = \sum_{n=0}^{\infty} v_{nz}(x) \quad \theta_m^n = \sum_{n=0}^{\infty} K_n(x) \theta_m^n. \tag{27}$$

By substituting Equations (25), (26), and (27) into (22), (23), and (24), and for small angles ($\sin \cong \theta, \cos\theta \cong 1$), we have

$$\begin{aligned}
&\rho i\omega_0 \sum_{n=0}^{\infty} v_{nx}\theta_m^n + \rho \sum_{n=0}^{\infty} v_{ny}\theta_m^n \frac{\partial}{\partial y}\sum_{n=0}^{\infty} v_{nx}\theta_m^n + \rho \sum_{n=0}^{\infty} v_{nz}\theta_m^n \frac{\partial}{\partial z}\sum_{n=0}^{\infty} v_{nx}\theta_m^n \\
&= \frac{1}{2}(\alpha_2+\alpha_4+\alpha_5)\frac{\partial^2}{\partial y^2}\sum_{n=0}^{\infty} v_{nx}\theta_m^n + \frac{\alpha_4}{2}\frac{\partial^2}{\partial z^2}\sum_{n=0}^{\infty} v_{nx}\theta_m^n \\
&\quad + \frac{1}{2}(\alpha_2+\alpha_5)\theta\frac{\partial^2}{\partial y \partial z}\sum_{n=0}^{\infty} v_{nx}\theta_m^n + \frac{1}{2}(\alpha_5-\alpha_2)\theta\frac{\partial^2}{\partial z \partial x}\sum_{n=0}^{\infty} v_{ny}\theta_m^n \\
&\quad + \frac{1}{2}\frac{\partial\theta}{\partial z}\left[(\alpha_2+\alpha_5)\frac{\partial}{\partial y}\sum_{n=0}^{\infty} v_{nx}\theta_m^n + (\alpha_5-\alpha_2)\frac{\partial}{\partial x}\sum_{n=0}^{\infty} v_{ny}\theta_m^n\right],
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\rho i\omega_0 \sum_{n=0}^{\infty} v_{ny}\theta_m^n + \rho \sum_{n=0}^{\infty} v_{nx}\theta_m^n \frac{\partial}{\partial x}\sum_{n=0}^{\infty} v_{ny}\theta_m^n + \rho \sum_{n=0}^{\infty} v_{nz}\theta_m^n \frac{\partial}{\partial z}\sum_{n=0}^{\infty} v_{ny}\theta_m^n \\
&= \frac{(\alpha_2+\alpha_4+\alpha_5)}{2}\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} v_{ny}\theta_m^n + \frac{1}{2}(\alpha_2+\alpha_4+\alpha_5)\frac{\partial^2}{\partial z^2}\sum_{n=0}^{\infty} v_{ny}\theta_m^n \\
&\quad + \frac{1}{2}(\alpha_2+\alpha_5)\theta\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} v_{nz}\theta_m^n,
\end{aligned} \tag{29}$$

$$\begin{aligned}
&\rho i\omega_0 \sum_{n=0}^{\infty} v_{nz}\theta_m^n + \rho \sum_{n=0}^{\infty} v_{nx}\theta_m^n \frac{\partial}{\partial x}\sum_{n=0}^{\infty} v_{nz}\theta_m^n \\
&= \frac{1}{2}\frac{\alpha_4\partial^2}{\partial x^2}\sum_{n=0}^{\infty} v_{nz}\theta_m^n + \frac{1}{2}(\alpha_2+\alpha_5)\theta\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} v_{ny}\theta_m^n + \alpha_5\theta\frac{\partial^2}{\partial z^2}\sum_{n=0}^{\infty} v_{ny}\theta_m^n \\
&\quad + \alpha_5\frac{\partial\theta}{\partial z}\frac{\partial}{\partial z}\sum_{n=0}^{\infty} v_{ny}\theta_m^n,
\end{aligned} \tag{30}$$

The zero order terms of Equations (28), (29), and (30) give

$$\rho i\omega_0 f_0(y)g_0(z) + \rho L_0(x)M_0(z)g_0(z)\frac{df_0(y)}{dy} + \rho K_0(x) f_0(y)\frac{dg_0(z)}{dz}$$
$$= \frac{(\alpha_2+\alpha_4+\alpha_5)}{2}g_0(z)\frac{d^2f_0(y)}{dy^2} + \frac{\alpha_4}{2}f_0(y)\frac{d^2g_0(z)}{dz^2}, \tag{31}$$

$$\rho i\omega_0 L_0(x)M_0(z) + \rho f_0(y)g_0(z)M_0(z)\frac{dL_0(x)}{dx} + \rho K_0(x)L_0(x)\frac{dM_0(z)}{dz}$$
$$= \frac{(\alpha_2+\alpha_4+\alpha_5)}{2}\left[M_0(z)\frac{d^2L_0(x)}{dx^2} + L_0(x)\frac{d^2M_0(z)}{dz^2}\right], \tag{32}$$

$$\rho i\omega_0 K_0(x) + \rho f_0(y)g_0(z)\frac{dK_0(x)}{dx} = \frac{\alpha_4}{2}\frac{d^2K_0(x)}{dx^2}. \tag{33}$$

The righthand side of Equation (33) is the only function of variable x, whereas the lefthand side of it is the function of all three variables x, y, and z. To overcome this difficulty one may assume $f_0(y)g_0(z) = c_0 =$ constant. The constant c_0 can be determined from Equation (31), and so

$$\rho i\omega c_0 = 0$$

or

$$v_{0x}(y,z) = c_0 = 0. \tag{34}$$

Hence Equation (33) gives

$$\frac{d^2K_0(x)}{dx^2} - \gamma_1^2 K_0(x) = 0, \tag{35}$$

where $\gamma_1^2 = \frac{2\rho i\omega_0}{\alpha_4}$. The solution of above equation is

$$K_0(x) = A\sinh(\gamma_1 x) + B\cosh(\gamma_1 x). \tag{36}$$

The constants A and B can be obtained by boundary condition

$$\bar{v} = r\omega\hat{\phi} = -\omega y\hat{i} + \omega x\hat{j}$$

at

$$z = \pm d/2.$$

Thus we have

$$v_0(x) = K_0(x) = 0. \tag{37}$$

By substituting Equations (34) and (37) into Equation (32), we may write

$$\frac{1}{L_0(x)}\frac{d^2L_0(x)}{dx^2}+\frac{1}{M_0(z)}\frac{d^2M_0(z)}{dz^2}-\gamma^2=0, \tag{38}$$

where $\gamma^2=\dfrac{2\rho i\omega_0}{\alpha_2+\alpha_4+\alpha_5}$.

By using the boundary condition on $L_0(x)$, ($L_0(x)=x$), and from Equation (38) we have

$$v_{0y}(x,y)=L_0(x)M_0(z)=\frac{x\omega}{\cosh(\gamma d/2)}\cosh(\gamma z). \tag{39}$$

The first-order terms of Equations (28), (29), and (30) give

$$\begin{aligned}\rho i\omega_0 f_1(y)g_1(z)&+\frac{\rho x\omega}{\cosh(\gamma d/2)}\cosh(\gamma z)g_1(z)\frac{df_1(y)}{dy}=\frac{(\alpha_2+\alpha_4+\alpha_5)}{2}\\ &\times g_1(z)\frac{d^2f_1(y)}{dy^2}+\frac{\alpha_4}{2}f_1(y)\frac{d^2g_1(z)}{dz^2}+\frac{(\alpha_5-\alpha_2)\gamma\omega}{2\cosh(\gamma d/2}\cos(\pi z/d)\sinh(\gamma z)\\ &-\frac{(\alpha_5-\alpha_2)\pi\omega}{2d\cosh(\gamma d/2)}\sin(\pi z/d)\cosh(\gamma z),\end{aligned} \tag{40}$$

$$\begin{aligned}\rho i\omega_0 L_1(x)M_1(z)&+\frac{\rho\omega}{\cosh(\gamma d/2)}f_1(y)g_1(z)\cosh(\gamma z)+\frac{\rho x\gamma\omega}{\cosh(\gamma d/2)}\sinh(\gamma \mathrm{z})\\ &=\frac{(\alpha_2+\alpha_4+\alpha_5)}{2}\left[M_1(z)\frac{d^2L_1(x)}{dx^2}+L_1(x)\frac{d^2M_1(z)}{dz^2}\right],\end{aligned} \tag{41}$$

$$\begin{aligned}\rho i\omega_0 K_1(x)&=\frac{\alpha_4}{2}\frac{d^2K_1(x)}{dx^2}+\frac{\alpha_5 x\gamma^2\omega}{\cosh(\gamma d/2)}\cos(\pi z/d)\cosh(\gamma z)\\ &-\frac{\alpha_5 x\pi\gamma\omega}{d\cosh(\gamma d/2)}\sin(\pi z/d)\sinh(\gamma z).\end{aligned} \tag{42}$$

The lefthand side of Equation (42) is the function of variable x only, whereas the righthand side of it is the function of variables x and z. To overcome this difficulty we see that average of second and third terms of the righthand side of Equation (42) are zero, i.e.,

$$\frac{\alpha_5 x\gamma\omega}{d\cosh(\gamma d/2)}\int_{d/2}^{-d/2}[\gamma\cos(\pi z/d)\cosh(\gamma z)-(\pi/d)\sin(\pi z/d)\sinh(\gamma z)]dz=0. \tag{43}$$

Hence, Equation (42) may be written as

$$\frac{d^2K_1(x)}{dx^2} - \gamma_1^2 K_1(x) = 0. \tag{44}$$

The solution of Equation (44) is similar to Equation (35):

$$v_{1z}(x) = K_1(x) = 0. \tag{45}$$

For the solution of Equation (40), on the bases of the problem symmetry one may assume that $f_1(y) = 1$.
Thus we have

$$\frac{d^2g_1(z)}{dz^2} - \gamma_1^2 g_1(z) + \frac{(\alpha_5 - \alpha_2)\omega}{\alpha_4 \cosh(\gamma d/2)}[\gamma\cos(\pi z/d)\sinh(\gamma z) - (\pi/d)\sin(\pi z/d)\cosh(\gamma z)] = 0 \tag{46}$$

The solution of Equation (46) for MBBA ($\gamma_1 d \cong 10^{-3}, \gamma d \cong 10^{-3}$ and $\gamma z \cong 10^{-3}$), which is typical for nematic LC, is

$$g_1(z) = \frac{(\alpha_5 - \alpha_2)\pi\omega}{\alpha_4}(2z - d\sin(\pi z/d)). \tag{48}$$

Thus, by using the boundary condition on $L_1(x), (L_1(x) = 1)$, and form Equation (41) we have

$$\frac{d^2M_1(z)}{dz^2} - \gamma^2 M_1(z) + \frac{2\rho(\alpha_2 - \alpha_5)\pi\omega^2}{\alpha_4(\alpha_2 + \alpha_4 + \alpha_5)\cosh(\gamma d/2)}\cosh(\gamma z)(2z - d\sin(\pi z/d)) = 0. \tag{49}$$

We now turn to solve the following equation:

$$\frac{d^2M_n(z)}{dz^2} - \gamma^2 M_n(z) + \frac{\rho(n+1)(\alpha_2 - \alpha_5)\pi\omega^2}{\alpha_4(\alpha_2 + \alpha_4 + \alpha_5)\cosh(\gamma d/2)}\cosh(\gamma z)(2z - d\sin(\pi z/d)) = 0. \tag{50}$$

The solution of Equation (50) for MBBA ($\gamma_1 d \cong 10^{-3}, \gamma d \cong 10^{-3}$ and $\gamma z \cong 10^{-3}$) is

$$M_n(z) = \left(\frac{n+1}{2}\right)Qz, \quad n = (1, 3, 5, ...), \tag{51}$$

where

$$Q = \frac{\rho d(\alpha_2 - \alpha_5)(1 - \pi)\omega^2}{\alpha_4(\alpha_2 + \alpha_4 + \alpha_5)}. \tag{52}$$

The solution of Equation (49) is similar to Equation (50), so we may write

$$v_{1y}(x,z) = L_1(x)M_1(z) = Qz. \tag{53}$$

The second-order terms of Equation (28), (29), and (30) give

$$\rho i\omega_0 f_2(y)g_2(z) + \frac{\rho x\omega}{\cosh(\gamma d/2)}M_2(z)\cosh(\gamma z)\frac{df_2(y)}{dy} = \frac{(\alpha_2+\alpha_4+\alpha_5)}{2}g_2(z)\frac{d^2f_2(y)}{dy^2} + \frac{\alpha_4}{2}f_2(y)\frac{d^2g_2(z)}{dz^2}, \tag{54}$$

$$\rho i\omega_0 L_2(x)M_2(z) + \frac{\rho\omega}{\cosh(\gamma d/2)}f_2(y)g_2(z) + \frac{\rho x\gamma\omega}{\cosh(\gamma d/2)}\sinh(\gamma z)K_2(x) = \frac{(\alpha_2+\alpha_4+\alpha_5)}{2}\left[M_2(z)\frac{d^2L_2(x)}{dx^2} + L_2(x)\frac{d^2M_2(z)}{dz^2}\right], \tag{55}$$

$$\rho i\omega_0 K_2(x) = \frac{\alpha_4}{2}\frac{d^2K_2(x)}{dx^2} - \frac{\alpha_5\pi Q}{d}\sin(\pi z/d). \tag{56}$$

Following the same argument that we solve Equation (42), we may write

$$\frac{d^2K_2(x)}{dx^2} - \gamma_1^2K_2(x) = 0 \tag{57}$$

and

$$v_{2z}(x) = K_2(x) = 0. \tag{58}$$

For the solution of Equation (54), we assume again that $f_2(y) = 1$. Thus, we have

$$\frac{d^2g_2(z)}{dz^2} - \gamma_1^2 g_2(z) = 0. \tag{59}$$

By using the boundary condition at $z = \pm d/2$ and Equation (59), we have

$$v_{2x}(y,z) = f_2(y)g_2(z) = 0. \tag{60}$$

By using Equations (58) and (60) in Equation (55), we may write

$$\frac{1}{L_2(x)}\frac{d^2L_2(x)}{dx^2} + \frac{1}{M_2(z)}\frac{d^2M_2(z)}{dz^2} - \gamma^2 = 0. \tag{61}$$

The solution of Equation (61) is similar to Equation (38). Thus, we may write

$$v_{2y}(x,z) = L_2(x)M_2(z) = \frac{x\omega}{\cosh(\gamma d/2)}\cosh(\gamma z) \tag{62}$$

The third-order terms of Equations (28), (29), and (30) give

$$\rho i\omega_0 f_3(y)g_3(z) + \frac{\rho x\omega}{\cosh(\gamma d/2)}\cosh(\gamma z)g_3(z)\frac{df_3(y)}{dy}$$
$$= \frac{1}{2}(\alpha_2+\alpha_4+\alpha_5)g_3(z)\frac{d^2f_3(y)}{dy^2} + \frac{\alpha_4}{2}f_3(y)\frac{d^2g_3(z)}{dz^2}$$
$$+ \frac{(\alpha_5-\alpha_2)\gamma\omega}{2\cosh(\gamma d/2)}\cos(\pi z/d)\sinh(\gamma z)$$
$$- \frac{(\alpha_5-\alpha_2)\pi\omega}{2d\cosh(\gamma d/2)}\sin(\pi z/d)\cosh(\gamma z), \tag{63}$$

$$\rho i\omega_0 L_3(x)M_3(z) + \frac{\rho\omega}{\cosh(\gamma d/2)}f_1(y)g_1(z)\cosh(\gamma z)$$
$$+ \frac{\rho\omega}{\cosh(\gamma d/2)}L_3(x)M_3(z)\cosh(\gamma z)$$
$$+ \frac{\rho x\gamma\omega}{\cosh(\gamma d/2)}K_3(x)\sinh(\gamma z)$$
$$= \frac{(\alpha_2+\alpha_4+\alpha_5)}{2}\left[M_3(z)\frac{d^2L_3(x)}{dx^2} + L_3(x)\frac{d^3M_3(z)}{dz^2}\right], \tag{64}$$

$$\rho i\omega_0 K_3(x) = \frac{\alpha_4}{2}\frac{d^2K_3(x)}{dx^2} + \frac{\alpha_5 x\gamma^2\omega}{\cosh(\gamma d/2)}\cos(\pi d/z)\cosh(\gamma z)$$
$$- \frac{\alpha_5 x\pi\gamma\omega}{d\cosh(\gamma d/2)}\sin(\pi z/d)\sinh(\gamma z). \tag{65}$$

The solution of Equation (65) is similar to Equation (42). Thus, we may write

$$v_{3z}(x) = K_3(x) = 0. \tag{66}$$

The solution of Equation (64) is similar to Equations (40), and (50) we may write

$$v_{3x}(y,z) = f_3(y)g_3(z) = \frac{\pi(\alpha_5-\alpha_2)\omega}{\alpha_4}(2z - d\sin(\pi z/d)). \tag{67}$$

By using the boundary condition on $L_3(x)$, ($L_3(x) = 1$), and form Equation (64) we have

$$\frac{d^2M_3(z)}{dz^2} - \gamma^2 M_3(z) + \frac{4\rho\pi(\alpha_2-\alpha_5)\omega^2}{\alpha_4(\alpha_2+\alpha_4+\alpha_5)\cosh(\gamma d/2)}$$
$$\times\cosh(\gamma z)(2z - d\sin(\pi z/d)) = 0 \tag{68}$$

The solution of Equation (68) is similar to Equation (50). Hence, we may write

$$v_{3y}(x, y) = L_3(x)M_3(z) = 2Qz. \tag{69}$$

Finally, by continuing this procedure the velocity field components may be written as

$$v_x(y, z) = \sum_{n=0}^{\infty} \frac{[1 + (-1)^{n+1}]\pi(\alpha_5 - \alpha_2)\omega}{2\alpha_4}(2z - d\sin(\pi z/d))\theta_m^n, \tag{70}$$

$$v_y(x, z) = \sum_{n=0}^{\infty} \frac{[1 + (-1)^n]x\omega}{2\cosh(\gamma d/2)}\cosh(\gamma z)\theta_m^n + \sum_{n=0}^{\infty} \frac{[1 + (-1)^{n+1}](n + 1)}{4}Qz\theta_m^n, \tag{71}$$

$$v_z(x) = 0. \tag{72}$$

CALCULATION OF THE TORQUE EXERTED ON THE SLAB

The torque exerted by the fluid on the oscillator is defined by

$$\Gamma_z = \rho\frac{d}{dt}\int (xv_y - yv_x)d^3x. \tag{73}$$

Furthermore, we define $\Delta\Gamma_z = \Gamma_z(\theta) - \Gamma_z(0)$, which is the variation in the exerted torque due to the appearance of the texture. In the uniform texture ($B \leq B_e$), we have $\Delta\Gamma_z = 0$.
Straightforward calculations give

$$\Delta\Gamma_z = \frac{d\omega}{dt}\frac{a^4\pi\rho\tanh(\gamma d/2)}{2\gamma}\theta_m^2. \tag{74}$$

We write $\Delta\Gamma_z$ in terms of the dimensionless quantities Δf_1 and Δf_2 as [5]

$$\Delta\Gamma_z = \frac{d\omega}{dt}\frac{a^4\pi\rho\tanh(\gamma d/2)}{2\gamma}(\Delta f_1 + i\Delta f_2), \tag{75}$$

where Δf_1 and Δf_2 are related to the inertial and dissipative effects, respectively. Here we calculated Δf_1 and Δf_2 for the case in which γd is much smaller than one:

$$\Delta\Gamma_z = \pi\rho\frac{d\omega}{dt}a^4 d\left(\frac{\theta_m^2}{4}\right). \tag{76}$$

As is obvious, $\Delta f_2 = 0$ for this case.

CONCLUDING REMARKS

The hydrodynamic equations for a narrow slab when the slate of nematic liquid crystal is deformed slightly by a perpendicular magnetic field are solved for planar texture. The components of the velocity field are obtained in Equations (70)–(72). $\Delta\Gamma_z$ or the values of Δf_1 and Δf_2 are proportional to θ_m^2. Δf_1 is proportional to the change in the resonant frequency and Δf_2 to its width. As we have mentioned previously, for the special case in which γd is much smaller than one, $\Delta f_2 = 0$ and Δf_1 is given by

$$\Delta f_1 = \theta_m^2 \tag{77}$$

Therefore, resonante frequency is given by

$$v_R = v_0 - \frac{\pi v_0 \rho a^4 d}{4I}\theta_m^2, \tag{78}$$

where $v_0 = \frac{1}{2\pi}\left(\frac{c}{I}\right)^{\frac{1}{2}}$ is the resonant frequency of the empty cell, I is the moment of inertia of the empty oscillator, and c is the torsion constant. As is obvious, the resonant frequency of the torsional oscillator and its width are proportional to the θ_m^2, which is given in Equation (7). The torsional oscillator is thus another probe for seeing the Fredriks transition in the nematic LC.

It is noted that the hydrodynamic equation for a narrow slab of nematic LC is calculated by Kadivar [6]. In this work, the following stress tensor is assumed:

$$\sigma_{\alpha\beta} = \eta\left(\frac{\partial v_\beta}{\partial x_\alpha} + \frac{\partial v_\alpha}{\partial x_\beta}\right), \tag{79}$$

where η is the coefficient of viscosity and is written as [2]

$$\eta = (\eta_1 + \eta_2\cos^2\theta)\sin^2\theta\cos^2\theta + \eta_2\cos^2\theta + \eta_3\sin^2\theta\sin^2\phi. \tag{80}$$

It is interesting that within the mentioned approximations our result in Equation (77) is the same as the Kadivar's result.

de Gennes and Prost [1] and de Jeu [2] consider a torsional cylindrical nematic sample in the presence of an applied magnetic field. By assuming the ordered phase without any texture or reorientations, they could relate the torque exerted on the cylinder to the coefficients $(\alpha_3 - \alpha_2)$, whereas our procedure in this article is different from them, since it can be a probe for detecting the Frederiks transitions and obtaining some of the viscosity or elastic coefficients by determining the resonant frequency. We also obtain the velocity field in the reorientations phase.

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